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Systems with intensity-dependent conversion integrable by finite orthogonal polynomials

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Abstract

We present exact solutions of a class of the nonlinear models which describe the parametric conversion of photons. Hamiltonians of these models are related to the classes of finite orthogonal polynomials. The spectra and exact expressions for eigenvectors of these Hamiltonians are obtained.

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1. Introduction

In nonlinear optical models the influence of a medium on electromagnetic field $(\vec{E}, \vec{B})$ is described by the material source-free Maxwell equations, where in general a functional dependence $\vec{P} = \vec{P}[\vec{E}]$ of the polarization $\vec{P}$ on the electric field $\vec{E}$ is assumed (see, e.g., [P-L, H-O-T, B-C]). This dependence describes complicated microstructure of the medium and the nonlinearity of the matter–field interactions. Assuming the classical description of the medium, the field can be quantized, and the energy operator is obtained. In the case of the two-mode field the operator is given by

$$\mathbf{H} = \mathbf{H}_0 + \mathrm{e}^{-\mathrm{i}\mathbf{H}_0 t}\mathbf{H}_I\,\mathrm{e}^{\mathrm{i}\mathbf{H}_0 t} \tag{1.1}$$

where

$$\mathbf{H}_0 = \omega_0\mathbf{a}_0^*\mathbf{a}_0 + \omega_1\mathbf{a}_1^*\mathbf{a}_1 \tag{1.2}$$

describes the free field, and the term

$$\mathbf{H}_I = \sum_{k,l,m,n=0}^{\infty}\left(\alpha_{klmn}(\mathbf{a}_0^*)^k(\mathbf{a}_1^*)^l\mathbf{a}_0^m\mathbf{a}_1^n + \mathrm{h.c.}\right) \tag{1.3}$$

where $\alpha_{1010} = \alpha_{0101} = 0$ is the interaction Hamiltonian responsible for the light–matter interactions. The annihilation and creation operators of two modes $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_0^*, \mathbf{a}_1^*$ fulfil the Heisenberg canonical commutation relations. Using the boson-number ordering, see [O-H-T], $\mathbf{H}_I$ can be expressed in the form

$$\mathbf{H}_I = \left(\sum_{k_0,k_1=1}^{\infty} f_{k_0k_1}(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1)\mathbf{a}_0^{k_0}\mathbf{a}_1^{k_1} + \text{h.c.} \right) + \left(\sum_{k_0,k_1=0}^{\infty} g_{k_0k_1}(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1)\mathbf{a}_0^{k_0}(\mathbf{a}_1^*)^{k_1} + \text{h.c.} \right) \tag{1.4}$$

where $f_{k_0k_1}(x, y)$ and $g_{k_0k_1}(x, y)$ are functions of two arguments x, y. These functions are defined by the constants α_{klmn} and are responsible for the light–matter interaction via the functional dependence $\overrightarrow{P} = \overrightarrow{P}[\overrightarrow{E}]$. If the sum in (1.3) is finite then $f_{k_0k_1}(x, y)$ and $g_{k_0k_1}(x, y)$ are polynomials. This case has been investigated during the last decade by many authors, see, e.g., [G-K-O, J-D, K]. They used approximate or semiclassical methods.

Let us give the interpretation of the particular terms of the Hamiltonian (1.4). The term $g_{k_0k_1}(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1)\mathbf{a}_0^{k_0}(\mathbf{a}_1^*)^{k_1}$ describes the process of simultaneous absorption of k_0 photons in mode 0 and emission of k_1 photons in mode 1. The probability of this process depends on $g_{k_0k_1}(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1)$, i.e. it depends on the intensity of light in the medium. So the function $g_{k_0k_1}$ is a generalization of the coupling constant for the conversion. Such a process is called the intensity-dependent [J-D] (or parametric [Pe-Lu]) conversion of k_0 photons in mode 0 into k_1 photons in mode 1. The Hermitian conjugate term $\left(g_{k_0k_1}(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1)\mathbf{a}_0^{k_0}(\mathbf{a}_1^*)^{k_1}\right)^*$ describes the parametric conversion of k_1 photons in mode 1 into k_0 photons in mode 0 with coupling given by the operator $\overline{g}_{k_0k_1}(\mathbf{a}_0^*\mathbf{a}_0 + k_0, \mathbf{a}_1^*\mathbf{a}_1 - k_1)$. By analogy the term $f_{k_0k_1}(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1)\mathbf{a}_0^{k_0}\mathbf{a}_1^{k_1}$ corresponds to the process of absorption by the medium of the cluster consisting of k_0 photons in mode 0 and k_1 photons in mode 1. The Hermitian conjugate term describes the emission of the same cluster by the medium.

In this paper we study the intensity-dependent conversion of a fixed number k_0 of photons in mode 0 into a fixed number k_1 of photons in mode 1 and vice versa. To simplify the notation let $h_0 := g_{00} + \overline{g}_{00}$ and $g := g_{k_0k_1}$; the interaction Hamiltonian for such a process takes the form

$$\mathbf{H}_I = h_0(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1) + g(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1)\mathbf{a}_0^{k_0}(\mathbf{a}_1^*)^{k_1} + (\mathbf{a}_0^*)^{k_0}\mathbf{a}_1^{k_1}\overline{g}(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1) \tag{1.5}$$

with the complex-valued function

$$g(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1) = \mathrm{e}^{\mathrm{i}\theta(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1)}|g(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1)|. \tag{1.6}$$

This paper is a continuation of the programme initiated in [O-H-T, H-O-T], where the theory of orthogonal polynomials was applied to the solution of the eigenproblem for the Hamiltonians of the type (1.5) for general functions h_0 and g. The solution of the eigenproblem of the interaction Hamiltonian $\mathbf{H}_I$ is equivalent to the problem of the integrability of the system under consideration; it follows from the fact that the solution of the Schrödinger equation for the Hamiltonian (1.1) is

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\mathbf{H}_0 t}\,\mathrm{e}^{-\mathrm{i}\mathbf{H}_I t}|\psi(0)\rangle. \tag{1.7}$$

In section 2 a family of operators commuting with the Hamiltonians (1.2) and (1.3) is found, i.e. the integrals of motion of the system. It is shown that the Hilbert space can be split into finite-dimensional subspaces invariant under the action of $\mathbf{H}_0$ and $\mathbf{H}_I$. Introducing the Fock basis in the subspaces, the matrix form of the interaction Hamiltonian is the Jacobi matrix. In section 3 it is shown that the eigenproblem of the considered Hamiltonian is equivalent to the moment problem of the theory of finite orthogonal polynomials. Using this equivalence we express the spectral decomposition of the Hamiltonian in terms of the orthogonal polynomials.

Section 4 is devoted to the construction of the families of Hamiltonians related to the fixed systems of orthonormal polynomials. In section 5 the families of integrable Hamiltonians are presented. We present examples of the simplest Hamiltonians related to the well-known families of finite orthogonal polynomials. The spectra and eigenvectors of these Hamiltonians are presented.

2. Conversion of fixed numbers of photons

Let us show that the two mode Fock space $\mathcal{H}$ can be split into finite-dimensional subspaces, invariant with respect to the interaction Hamiltonian (1.5) and the free Hamiltonian (1.2), and therefore also invariant with respect to the Hamiltonian (1.1). First we observe that the following operators:

$$\mathbf{K} := k_1 \mathbf{a}_0^* \mathbf{a}_0 + k_0 \mathbf{a}_1^* \mathbf{a}_1 \tag{2.1}$$

$$\mathbf{R}_\kappa := \frac{k_\kappa - 1}{2} + \sum_{s=1}^{k_\kappa - 1} \frac{\exp\left(-\mathrm{i}\frac{2\pi s}{k_\kappa}\mathbf{a}_\kappa^* \mathbf{a}_\kappa\right)}{\exp\left(\mathrm{i}\frac{2\pi s}{k_\kappa}\right) - 1} \tag{2.2}$$

where $\kappa = 0, 1$, commute with $\mathbf{H}_0$ and $\mathbf{H}_I$, i.e. they are integrals of motion of the system. k_0 and k_1 are defined by (1.5). The elements of the Fock basis of $\mathcal{H}$

$$|n_0, n_1\rangle = \frac{1}{\sqrt{n_0! n_1!}} (\mathbf{a}_0^*)^{n_0} (\mathbf{a}_1^*)^{n_1} |0, 0\rangle \qquad n_0, n_1 = 0, 1, \ldots \tag{2.3}$$

are eigenvectors of the operators (2.1), (2.2)

$$\mathbf{K}|n_0, n_1\rangle = (k_1 n_0 + k_0 n_1)|n_0, n_1\rangle \tag{2.4}$$

$$\mathbf{R}_\kappa |n_0, n_1\rangle = r_\kappa |n_0, n_1\rangle \tag{2.5}$$

where the eigenvalues r_κ are equal to the remainder of the division of n_κ by k_κ ($r_\kappa = n_\kappa (\mathrm{mod}\, k_\kappa)$), see [G-K-O]. So one has the orthogonal decomposition

$$\mathcal{H} = \bigoplus_{\mu \in J} \mathcal{H}_\mu \tag{2.6}$$

of $\mathcal{H}$ into finite-dimensional Hilbert subspaces $\mathcal{H}_\mu$ labelled by multi-indices $\mu := (r_0, r_1, N) \in J := \{0, 1, \ldots, k_0 - 1\} \times \{0, 1, \ldots, k_1 - 1\} \times (\mathbb{N} \cup \{0\})$ and spanned by the vectors

$$|n\rangle_\mu := |r_0 + k_0 n, r_1 + k_1(N - n)\rangle \qquad n = 0, 1, \ldots, N \tag{2.7}$$

of the Fock basis. The Hilbert subspace $\mathcal{H}_\mu$ is a common eigenspace of the operators $\mathbf{K}$, $\mathbf{R}_0$ and $\mathbf{R}_1$ with dimension

$$\dim \mathcal{H}_\mu = N + 1. \tag{2.8}$$

Moreover, $N + 1$ can be obtained as the only eigenvalue of the dimension operator

$$\mathbf{D} = \frac{1}{k_0 k_1}\mathbf{K} - \frac{1}{k_0}\mathbf{R}_0 - \frac{1}{k_1}\mathbf{R}_1 + 1 \tag{2.9}$$

on $\mathcal{H}_\mu$. According to the above let us introduce the operators

$$\mathbf{A}_0 := \frac{1}{k_0}\mathbf{a}_0^* \mathbf{a}_0 \tag{2.10}$$

$$\mathbf{A} := g(\mathbf{a}_0^* \mathbf{a}_0, \mathbf{a}_1^* \mathbf{a}_1) \mathbf{a}_0^{k_0} (\mathbf{a}_1^*)^{k_1} \tag{2.11}$$

and also replace the operators $\mathbf{a}_0^* \mathbf{a}_0$ and $\mathbf{a}_1^* \mathbf{a}_1$ by $\mathbf{A}_0$ and $\mathbf{K}$.

The operators $\mathbf{A}_0, \mathbf{A}, \mathbf{A}^*$ satisfy the relations

$$[\mathbf{A}_0, \mathbf{A}] = -\mathbf{A} \qquad [\mathbf{A}_0, \mathbf{A}^*] = \mathbf{A}^* \qquad \mathbf{A}^*\mathbf{A} = \mathcal{G}(\mathbf{A}_0 - 1, \mathbf{K}) \qquad \mathbf{A}\mathbf{A}^* = \mathcal{G}(\mathbf{A}_0, \mathbf{K}) \tag{2.12}$$

where the function $\mathcal{G}(\mathbf{A}_0, \mathbf{K})$ is determined by g

$$\mathcal{G}(\mathbf{A}_0, \mathbf{K}) := \left| g\left(k_0\mathbf{A}_0, \frac{1}{k_0}\mathbf{K} - k_1\mathbf{A}_0\right)\right|^2 (k_0\mathbf{A}_0 + 1)(k_0\mathbf{A}_0 + 2)\cdots(k_0\mathbf{A}_0 + k_0) \times \left(\frac{1}{k_0}\mathbf{K} - k_1\mathbf{A}_0\right)\left(\frac{1}{k_0}\mathbf{K} - k_1\mathbf{A}_0 - 1\right)\cdots\left(\frac{1}{k_0}\mathbf{K} - k_1\mathbf{A}_0 - k_1 + 1\right). \tag{2.13}$$

It can be shown that

$$\mathbf{A}_0|n\rangle_\mu = \left(\frac{r_0}{k_0} + n\right)|n\rangle_\mu \qquad \mathbf{A}|n\rangle_\mu = b_{n-1,\mu}|n-1\rangle_\mu \qquad \mathbf{A}^*|n\rangle_\mu = \overline{b_{n,\mu}}|n+1\rangle_\mu \tag{2.14}$$

where

$$b_{n,\mu} = \mathrm{e}^{-\mathrm{i}\phi(\frac{r_0}{k_0}+n, k_1r_0+k_0r_1+k_0k_1N)}\sqrt{\mathcal{G}\left(\frac{r_0}{k_0} + n, k_1r_0 + k_0r_1 + k_0k_1N\right)} \tag{2.15}$$

i.e. $\mathbf{A}_0$ is diagonal, $\mathbf{A}$ and $\mathbf{A}^*$ are weighted shift operators and ϕ denotes the phase factor of function g expressed in the new variables n and N. Let us note that from (2.13) it follows that $\mathcal{G}(-1, N) = \mathcal{G}(N, N) = 0$ which makes (2.14) consistent.

The interaction Hamiltonian (1.5) now takes the form

$$\mathbf{H}_I = h(\mathbf{A}_0, \mathbf{K}) + \mathbf{A} + \mathbf{A}^* \tag{2.16}$$

where h is uniquely determined by h_0 and the free Hamiltonian $\mathbf{H}_0$ is given by

$$\mathbf{H}_0 = (\omega_0k_0 - \omega_1k_1)\mathbf{A}_0 + \frac{\omega_1}{k_0}\mathbf{K}. \tag{2.17}$$

Thus the Hamiltonian describing our system belongs to the operator algebra generated by the operators $\mathbf{K}, \mathbf{A}_0, \mathbf{A}, \mathbf{A}^*$ ($\mathbf{K}$ commutes with the others). The subspaces $\mathcal{H}_\mu$ are invariant subspaces of the operators $\mathbf{H}_0, \mathbf{H}_I$ and therefore of $\mathbf{H}$. Moreover, the action of these operators on elements of the Fock basis is

$$\mathbf{H}_0|n\rangle_\mu = (\omega_0r_0 + \omega_1r_1 + \omega_1k_1N + (\omega_0k_0 - \omega_1k_1)n)|n\rangle_\mu \tag{2.18}$$

and

$$\mathbf{H}_I|n\rangle_\mu = b_{n-1,\mu}|n-1\rangle_\mu + a_{n,\mu}|n\rangle_\mu + \overline{b_{n,\mu}}|n+1\rangle_\mu \tag{2.19}$$

where $b_{n,\mu}$ is given by (2.15), and

$$a_{n,\mu} = h\left(\frac{r_0}{k_0} + n, k_1r_0 + k_0r_1 + k_0k_1N\right). \tag{2.20}$$

It follows from (2.19) that the matrix form of the operator $\mathbf{H}_\mu := \mathbf{H}_I|_{\mathcal{H}_\mu}$ in the basis $\{|n\rangle_\mu\}_{n=0}^N$ of $\mathcal{H}_\mu$ is the Jacobi matrix (three diagonal and Hermitian matrix). This fact allows us to use the theory of orthogonal polynomials to solve the eigenproblem of $\mathbf{H}_\mu$ [O-H-T, H-O-T].

3. Integration and finite polynomials

The three-term formula (2.19) suggests that the eigenproblem of $\mathbf{H}_\mu$ is strictly connected with the theory of finite orthogonal polynomials. One can apply this theory under the additional assumption that $\mathbf{H}_\mu$ has $N+1$ different eigenvalues $\{E_{l,\mu}\}_{l=0}^N$ with the corresponding eigenvectors $\{|E_{l,\mu}\rangle\}_{l=0}^N$, i.e.

$$\mathbf{H}_\mu|E_{l,\mu}\rangle = E_{l,\mu}|E_{l,\mu}\rangle. \tag{3.1}$$

Then the spectral decomposition of the interaction Hamiltonian is

$$\mathbf{H}_I = \sum_{\mu\in J}\sum_{l=0}^N E_{l,\mu}\frac{|E_{l,\mu}\rangle\langle E_{l,\mu}|}{\langle E_{l,\mu}|E_{l,\mu}\rangle}. \tag{3.2}$$

If we decompose $|E_{l,\mu}\rangle$ in the Fock basis $\{|n\rangle_\mu\}_{n=0}^N$ of $\mathcal{H}_\mu$

$$|E_{l,\mu}\rangle = \sum_{n=0}^N P_n^\mu(E_{l,\mu})|n\rangle_\mu \tag{3.3}$$

then from (2.19), (3.1) and (3.3) it follows that the coefficients $P_n^\mu(E_{l,\mu})$ satisfy the three term identity:

$$E_{l,\mu}P_n^\mu(E_{l,\mu}) = \overline{b_{n-1,\mu}}P_{n-1}^\mu(E_{l,\mu})+a_{n,\mu}P_n^\mu(E_{l,\mu}) + b_{n,\mu}P_{n+1}^\mu(E_{l,\mu}) \qquad n,l = 0,1,\ldots,N \tag{3.4}$$

which can be considered as the recurrence relations for $P_n^\mu(E_{l,\mu})$ with the initial condition $P_0^\mu(E_{l,\mu}) \equiv 1$. Thus $P_n^\mu(E_{l,\mu})$ is a polynomial of degree n in the variable $E_{l,\mu}$. Since $\mathbf{H}_\mu$ is Hermitian, the set $\{|E_{l,\mu}\rangle\}_{l=0}^N$ forms an orthogonal basis in $\mathcal{H}_\mu$. The orthogonality relations for eigenvectors $|E_{l,\mu}\rangle$ imply the orthonormality relation in the set of polynomials $\left\{P_n^\mu(E_{l,\mu})\right\}_{n=0}^N$

$$\sum_{l=0}^N \overline{P_n^\mu(E_{l,\mu})}P_m^\mu(E_{l,\mu})\frac{1}{\langle E_{l,\mu}|E_{l,\mu}\rangle} = \delta_{mn} \tag{3.5}$$

this allows us to invert the formula (3.3)

$$|n\rangle_\mu = \sum_{l=0}^N \frac{1}{\langle E_{l,\mu}|E_{l,\mu}\rangle}\overline{P_n^\mu(E_{l,\mu})}|E_{l,\mu}\rangle. \tag{3.6}$$

In such a way we obtain a finite system of orthonormal polynomials $\left\{P_n^\mu\right\}_{n=0}^N$, with respect to the weight function $\frac{1}{\langle E_{l,\mu}|E_{l,\mu}\rangle}$, dependent on the discrete variables $\{E_{l,\mu}\}_{l=0}^N$.

Since the interaction Hamiltonian $\mathbf{H}_I$ preserves the decomposition (2.6), the unitary one-parameter group of Schrödinger evolution in the interaction picture $\mathrm{e}^{-\mathrm{i}\mathbf{H}_I t}$ (see (1.7)) preserves the decomposition of the identity

$$\mathbf{1} = \sum_{\mu\in J}\sum_{n=0}^N |n\rangle_\mu\,{}_\mu\langle n|. \tag{3.7}$$

Thus, using the spectral decomposition (3.2) and the orthogonal relations (3.5) the operator $\mathrm{e}^{-\mathrm{i}\mathbf{H}_I t}$ can be expressed in the form

$$\mathrm{e}^{-\mathrm{i}\mathbf{H}_I t} = \sum_{\mu\in J}\sum_{l=0}^N \mathrm{e}^{-\mathrm{i}E_{l,\mu}t}\frac{|E_{l,\mu}\rangle\langle E_{l,\mu}|}{\langle E_{l,\mu}|E_{l,\mu}\rangle}. \tag{3.8}$$

Its matrix elements

$$ {}_{\nu}\langle m|\, \mathrm{e}^{-\mathrm{i}\mathbf{H}_I t} |n\rangle_{\mu} = \delta_{NS}\delta_{r_0 q_0}\delta_{r_1 q_1} \sum_{l=0}^{N} \overline{P_m^{\mu}(E_{l,\mu})}\, P_n^{\mu}(E_{l,\mu}) \frac{\mathrm{e}^{-\mathrm{i}t E_{l,\mu}}}{\langle E_{l,\mu}|E_{l,\mu}\rangle} \tag{3.9} $$

where $\mu = (r_0, r_1, N)$ and $\nu = (q_0, q_1, S)$ are expressed in terms of the orthogonal polynomials. From (1.7) and (3.9) the time evolution of the expectation value of any quantum observable $\mathbf{X}$ in a normalized state $|\psi\rangle \in \mathcal{H}$ becomes

$$ \begin{aligned} \langle \mathbf{X}(t)\rangle_{\psi} &\equiv \langle\psi|\,\mathrm{e}^{\mathrm{i}\mathbf{H}_I t}\,\mathrm{e}^{\mathrm{i}\mathbf{H}_0 t}\mathbf{X}\,\mathrm{e}^{-\mathrm{i}\mathbf{H}_0 t}\,\mathrm{e}^{-\mathrm{i}\mathbf{H}_I t}|\psi\rangle = \sum_{\mu,\nu\in J}^{\infty}\sum_{m,r,l=0}^{N}\sum_{k,s,n=0}^{S} \langle\psi|m\rangle_{\mu}\,{}_{\mu}\langle r|\mathbf{X}|s\rangle_{\nu}\,{}_{\nu}\langle n|\psi\rangle \\ &\quad\times \mathrm{e}^{-\mathrm{i}t(\omega_0(q_0-r_0)+\omega_1(q_1-r_1)+\omega_1 k_1(S-N)+(\omega_0 k_0-\omega_1 k_1)(s-r))} \\ &\quad\times \frac{\mathrm{e}^{-\mathrm{i}t(E_{k,\nu}-E_{l,\mu})}}{\langle E_{l,\mu}|E_{l,\mu}\rangle\langle E_{k,\nu}|E_{k,\nu}\rangle} \overline{P_m^{\mu}(E_{l,\mu})}\;\overline{P_s^{\nu}(E_{k,\nu})}\,P_r^{\mu}(E_{l,\mu})\,P_n^{\nu}(E_{k,\nu}). \end{aligned} \tag{3.10} $$

Similarly matrix elements of $\mathbf{X}(t)$ are equal to

$$ \begin{aligned} {}_{\mu}\langle m|\mathbf{X}(t)|n\rangle_{\nu} &= \sum_{l,r=0}^{N}\sum_{k,s=0}^{S} {}_{\mu}\langle r|\mathbf{X}|s\rangle_{\nu}\,\mathrm{e}^{-\mathrm{i}t(\omega_0(q_0-r_0)+\omega_1(q_1-r_1)+\omega_1 k_1(S-N)+(\omega_0 k_0-\omega_1 k_1)(s-r))} \\ &\quad\times \frac{\mathrm{e}^{-\mathrm{i}t(E_{k,\nu}-E_{l,\mu})}}{\langle E_{l,\mu}|E_{l,\mu}\rangle\langle E_{k,\nu}|E_{k,\nu}\rangle} \overline{P_m^{\mu}(E_{l,\mu})}\;\overline{P_s^{\nu}(E_{k,\nu})}\,P_r^{\mu}(E_{l,\mu})\,P_n^{\nu}(E_{k,\nu}). \end{aligned} \tag{3.11} $$

4. Hamiltonians related to the same families of fixed orthonormal polynomials

It is natural that different Hamiltonians after reduction can lead to the same family of orthogonal polynomials. In this section we solve the inverse problem, i.e. how to construct different Hamiltonians which are related to the same family of orthogonal polynomials.

Let us consider the interaction Hamiltonian in the case when $k_0 = k_1 = 1$

$$ \mathbf{H}_I = h_0(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1) + g(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1)\mathbf{a}_0\mathbf{a}_1^* + \mathbf{a}_0^*\mathbf{a}_1\overline{g}(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1). \tag{4.1} $$

Here the multi-index μ describing the decomposition (2.6) is a single index $\mu \equiv N \in \mathbb{N}\cup\{0\}$. Moreover, all components of that decomposition have different dimensions. Let us assume that the eigenproblem is solved by a family of orthonormal finite polynomials $\left\{P_n^N(E_{l,N})\right\}_{n=0}^{N}$. We will call (4.1) the initial Hamiltonian.

For any fixed $k_0, k_1 \in \mathbb{N}$ we construct the Hamiltonian

$$ \widetilde{\mathbf{H}}_I = \widetilde{h}_0(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1) + \widetilde{g}(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1)\mathbf{a}_0^{k_0}(\mathbf{a}_1^*)^{k_1} + (\mathbf{a}_0^*)^{k_0}\mathbf{a}_1^{k_1}\overline{\widetilde{g}}(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1) \tag{4.2} $$

in such a way that the eigenproblem for any $\widetilde{\mathbf{H}}_\mu$, where $\mu = (r_0, r_1, N) \in J = \{0, 1, \ldots, k_0-1\}\times\{0, 1, \ldots, k_1-1\}\times(\mathbb{N}\cup\{0\})$ is solved by the initial family $\left\{P_n^N(E_{l,N})\right\}_{n=0}^{N}$, i.e.

$$ \forall\mu = (r_0, r_1, N) \in J \qquad P_n^{\mu} = P_n^N \qquad \text{and} \qquad E_{l,\mu} = E_{l,N} \qquad n, l = 0, 1, \ldots, N. \tag{4.3} $$

In other words, the solution of the eigenproblem for $\mathbf{H}_\mu$ does not depend on r_0 and r_1.

Let us introduce the following operator:

$$ \begin{aligned} W_{k_0k_1}(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1) &:= \sqrt{(\mathbf{a}_0^*\mathbf{a}_0 - \mathbf{R}_0 + k_0)(\mathbf{a}_1^*\mathbf{a}_1 - \mathbf{R}_1)} \\ &\quad\times \frac{1}{\sqrt{k_0k_1(\mathbf{a}_0^*\mathbf{a}_0+1)(\mathbf{a}_0^*\mathbf{a}_0+2)\cdots(\mathbf{a}_0^*\mathbf{a}_0+k_0)\mathbf{a}_1^*\mathbf{a}_1(\mathbf{a}_1^*\mathbf{a}_1-1)\cdots(\mathbf{a}_1^*\mathbf{a}_1-k_1+1)}}. \end{aligned} \tag{4.4} $$

In particular $W_{11}(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1) \equiv \mathbf{1}$. It is easy to check that

$$W_{k_0k_1}(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1)\mathbf{a}_0^{k_0}(\mathbf{a}_1^*)^{k_1}|n\rangle_\mu = \sqrt{n(N-n+1)}|n-1\rangle_\mu. \tag{4.5}$$

Since additionally we have that

$$\frac{1}{k_0}(\mathbf{a}_0^*\mathbf{a}_0 - \mathbf{R}_0)|n\rangle_\mu = n|n\rangle_\mu \tag{4.6}$$

$$\frac{1}{k_1}(\mathbf{a}_1^*\mathbf{a}_1 - \mathbf{R}_1)|n\rangle_\mu = (N-n)|n\rangle_\mu \tag{4.7}$$

then putting

$$\widetilde{g}(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1) = g\left(\frac{1}{k_0}(\mathbf{a}_0^*\mathbf{a}_0 - \mathbf{R}_0), \frac{1}{k_1}(\mathbf{a}_1^*\mathbf{a}_1 - \mathbf{R}_1)\right) W_{k_0k_1}(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1) \tag{4.8}$$

and

$$\widetilde{h}_0(\mathbf{a}_0^*\mathbf{a}_0, \mathbf{a}_1^*\mathbf{a}_1) = h_0\left(\frac{1}{k_0}(\mathbf{a}_0^*\mathbf{a}_0 - \mathbf{R}_0), \frac{1}{k_1}(\mathbf{a}_1^*\mathbf{a}_1 - \mathbf{R}_1)\right) \tag{4.9}$$

we obtain the Hamiltonian (4.2) which satisfies conditions (4.3).

In the following section we present some examples of initial Hamiltonians which are related to the families of the finite discrete orthonormal polynomials, which are best known in the literature.

5. Quantum systems related to some classes of finite orthonormal polynomials

In this section we present a list of Hamiltonians for which the spectral decompositions can be expressed by some selected families of finite orthogonal polynomials. The results of the previous section allow us to restrict our list to the cases when $k_0 = k_1 = 1$. The notation used in this section is the same as in [K-S]. The hypergeometric series are denoted by

$${}_rF_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}\middle| z\right) := \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r)_n}{(b_1, \ldots, b_s)_n} \frac{z^n}{n!} \tag{5.1}$$

where $(a_1, \ldots, a_r)_n := (a_1)_n \cdots (a_r)_n$, $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$ for $n = 1, 2, \ldots$ and $(a)_0 := 1$. The basic hypergeometric series is defined by

$${}_r\phi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}\middle| q; z\right) := \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} (-1)^{(1+s-r)n} q^{(1+s-r)\binom{n}{2}} \frac{z^n}{(q;q)_n} \tag{5.2}$$

for $0 < q < 1$, where $(a_1, \ldots, a_r; q)_n := (a_1; q)_n \cdots (a_r; q)_n$ and $(a; q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$, for $n = 1, 2, \ldots$, and $(a; q)_0 = 1$.

5.1. Integrable systems related to the Krawtchouk polynomials

The Krawtchouk polynomials arise for a system described by the Hamiltonian

$$\mathbf{H}_I = p\mathbf{a}_1^*\mathbf{a}_1 + (1-p)\mathbf{a}_0^*\mathbf{a}_0 + \sqrt{p(1-p)}(\mathbf{a}_0\mathbf{a}_1^* + \mathbf{a}_0^*\mathbf{a}_1) \tag{5.3}$$

where $0 < p < 1$. The spectrum of $\mathbf{H}_I$ is

$$\sigma(\mathbf{H}_I) = \mathbb{N} \cup \{0\} \tag{5.4}$$

which follows from the fact that the eigenvalues of the reduced Hamiltonian $\mathbf{H}_N$ are $E_{l,N} = l$ for each N. Thus the eigenspaces $\mathcal{H}^l$, $l \in \sigma(\mathbf{H}_I)$ are the infinite-dimensional Hilbert subspaces

$$\mathcal{H}^l = \text{span}\{|E_{l,N}\rangle : N = l+n, n = 0, 1, \ldots\} \tag{5.5}$$

where

$$|E_{l,N}\rangle = \sum_{n=0}^{N} K_n(E_{l,N};\, p, N)|n, N-n\rangle \tag{5.6}$$

and

$$K_n(E_{l,N};\, p, N) = \sqrt{\frac{(-N)_n}{(-1)^n n!}\left(\frac{p}{1-p}\right)^n}\, {}_2F_1\left(\left.\begin{matrix}-n, -l\\ -N\end{matrix}\right|\frac{1}{p}\right) \tag{5.7}$$

are the Krawtchouk polynomials. For each fixed N, due to (3.5), the finite family $\{K_n(E_{l,N};\, p, N)\}_{n=0}^{N}$ forms an orthonormal system with respect to the weight function

$$\frac{1}{\langle E_{l,N}|E_{l,N}\rangle} = \binom{N}{l} p^l (1-p)^{N-l}. \tag{5.8}$$

We can summarize that the spectral decomposition of the interaction Hamiltonian is

$$\mathbf{H}_I = \sum_{N=0}^{\infty}\sum_{l=0}^{N} l\,\frac{|E_{l,N}\rangle\langle E_{l,N}|}{\langle E_{l,N}|E_{l,N}\rangle}. \tag{5.9}$$

The Hamiltonian (5.3) is quadratic in annihilation and creation operators and therefore the system is also integrable via Heisenberg equations.

5.2. Integrable systems related to the dual Hahn polynomials

The dual Hahn polynomials are related to the system given by the Hamiltonian

$$\mathbf{H}_I = \mathbf{a}_0^*\mathbf{a}_0(\mathbf{a}_1^*\mathbf{a}_1 + \delta + 1) + (\mathbf{a}_0^*\mathbf{a}_0 + \gamma + 1)\mathbf{a}_1^*\mathbf{a}_1 + \sqrt{(\mathbf{a}_1^*\mathbf{a}_1 + \delta)(\mathbf{a}_0^*\mathbf{a}_0 + \gamma + 1)}\,\mathbf{a}_0\mathbf{a}_1^* + \mathbf{a}_0^*\mathbf{a}_1\sqrt{(\mathbf{a}_1^*\mathbf{a}_1 + \delta)(\mathbf{a}_0^*\mathbf{a}_0 + \gamma + 1)} \tag{5.10}$$

where $\gamma, \delta > -1$. The spectrum of this Hamiltonian is

$$\sigma(\mathbf{H}_I) = \{l(l + \gamma + \delta + 1) : l = 0, 1, \ldots\} \tag{5.11}$$

and the eigenvalues of the reduced Hamiltonian $\mathbf{H}_N$ are $E_{l,N} = l(l + \gamma + \delta + 1)$ and do not depend on N. For each eigenvalue $l(l + \gamma + \delta + 1)$ the corresponding eigenspace $\mathcal{H}^l$ of $\mathbf{H}_I$ is the infinite-dimensional Hilbert space

$$\mathcal{H}^l = \operatorname{span}\{|E_{l,N}\rangle : N = l + n, n = 0, 1, \ldots\} \tag{5.12}$$

where

$$|E_{l,N}\rangle = \sum_{n=0}^{N} R_n(E_{l,N};\, \gamma, \delta, N)|n, N-n\rangle \tag{5.13}$$

and

$$R_n(E_{l,N};\, \gamma, \delta, N) = \sqrt{\binom{\gamma+n}{n}\binom{\delta+N-n}{N-n}}\, {}_3F_2\left(\left.\begin{matrix}-n, -l, l+\gamma+\delta+1\\ \gamma+1, -N\end{matrix}\right|1\right) \tag{5.14}$$

are the dual Hahn polynomials. The finite family $\{R_n(E_{l,N};\, \gamma, \delta, N)\}_{n=0}^{N}$, due to (3.5), forms an orthonormal system with respect to the weight function

$$\frac{1}{\langle E_{l,N}|E_{l,N}\rangle} = \frac{(2l + \gamma + \delta + 1)(\gamma + 1)_l(-1)^l(-N)_l N!}{(l + \gamma + \delta + 1)_{N+1}(\delta + 1)_l\, l!}. \tag{5.15}$$

We can summarize that the spectral decomposition of the interaction Hamiltonian is

$$\mathbf{H}_I = \sum_{N=0}^{\infty}\sum_{l=0}^{N} l(l + \gamma + \delta + 1)\frac{|E_{l,N}\rangle\langle E_{l,N}|}{\langle E_{l,N}|E_{l,N}\rangle}. \tag{5.16}$$

5.3. Integrable systems related to the discrete Chebyshev polynomials

The Hamiltonian

$$\mathbf{H}_I = \frac{(2\mathbf{a}_0^*\mathbf{a}_0 + \mathbf{a}_1^*\mathbf{a}_1 + 1)\mathbf{a}_0^*\mathbf{a}_0}{2(2\mathbf{a}_0^*\mathbf{a}_0 + 1)} + \frac{(\mathbf{a}_0^*\mathbf{a}_0 + 1)\mathbf{a}_1^*\mathbf{a}_1}{2(2\mathbf{a}_0^*\mathbf{a}_0 + 1)} + \frac{1}{2}\sqrt{\frac{(2\mathbf{a}_0^*\mathbf{a}_0 + \mathbf{a}_1^*\mathbf{a}_1 + 2)(\mathbf{a}_0^*\mathbf{a}_0 + 1)}{(2\mathbf{a}_0^*\mathbf{a}_0 + 1)(2\mathbf{a}_0^*\mathbf{a}_0 + 3)}}\mathbf{a}_0\mathbf{a}_1^* + \mathbf{a}_0^*\mathbf{a}_1\frac{1}{2}\sqrt{\frac{(2\mathbf{a}_0^*\mathbf{a}_0 + \mathbf{a}_1^*\mathbf{a}_1 + 2)(\mathbf{a}_0^*\mathbf{a}_0 + 1)}{(2\mathbf{a}_0^*\mathbf{a}_0 + 1)(2\mathbf{a}_0^*\mathbf{a}_0 + 3)}} \tag{5.17}$$

describes the system for which the solution of the eigenproblem is given by discrete Chebyshev polynomials. The spectrum of (5.17) is

$$\sigma(\mathbf{H}_I) = \mathbb{N} \cup \{0\} \tag{5.18}$$

and similarly as in section 5.1, the infinite-dimensional eigenspaces $\mathcal{H}^l$ of $\mathbf{H}_I$ are spanned by the vectors

$$|E_{l,N}\rangle = \sum_{n=0}^{N} T_n(E_{l,N}; N)|n, N-n\rangle \qquad N = l+n, n = 0, 1, \ldots \tag{5.19}$$

with the eigenvalues $E_{l,N} = l, l = 0, 1, \ldots$ of the reduced Hamiltonian $\mathbf{H}_N$. The coefficients

$$T_n(E_{l,N}; N) = \sqrt{\frac{(2n+1)(-N)_n N!}{(-1)^n(n+1)_{N+1}n!}}\, {}_3F_2\left(\left.\begin{matrix}-n, n+1, -l, \\ 1, -N\end{matrix}\right| 1\right) \tag{5.20}$$

are the discrete Chebyshev polynomials. The weight function for the family $\{T_n(E_{l,N}; N)\}_{n=0}^N$ is, as for classical Chebyshev polynomials, constant

$$\frac{1}{\langle E_{l,N}|E_{l,N}\rangle} \equiv 1. \tag{5.21}$$

We can summarize that the spectral decomposition of the interaction Hamiltonian is

$$\mathbf{H}_I = \sum_{N=0}^{\infty}\sum_{l=0}^{N} l|E_{l,N}\rangle\langle E_{l,N}|. \tag{5.22}$$

5.4. Integrable systems related to the Hahn polynomials

The Hahn polynomials arise for the system described by the Hamiltonian

$$\begin{aligned}\mathbf{H}_I = {} & \frac{\mathbf{a}_0^*\mathbf{a}_0(2\mathbf{a}_0^*\mathbf{a}_0 + \mathbf{a}_1^*\mathbf{a}_1 + \alpha + \beta + 1)(\mathbf{a}_0^*\mathbf{a}_0 + \beta)}{(2\mathbf{a}_0^*\mathbf{a}_0 + \alpha + \beta)(2\mathbf{a}_0^*\mathbf{a}_0 + \alpha + \beta + 1)} \\ & + \frac{(\mathbf{a}_0^*\mathbf{a}_0 + \alpha + 1)(\mathbf{a}_0^*\mathbf{a}_0 + \alpha + \beta + 1)\mathbf{a}_1^*\mathbf{a}_1}{(2\mathbf{a}_0^*\mathbf{a}_0 + \alpha + \beta + 1)(2\mathbf{a}_0^*\mathbf{a}_0 + \alpha + \beta + 2)} \\ & + \sqrt{\frac{(2\mathbf{a}_0^*\mathbf{a}_0 + \mathbf{a}_1^*\mathbf{a}_1 + \alpha + \beta + 2)(\mathbf{a}_0^*\mathbf{a}_0 + \beta + 1)(\mathbf{a}_0^*\mathbf{a}_0 + \alpha + 1)(\mathbf{a}_0^*\mathbf{a}_0 + \alpha + \beta + 1)}{(2\mathbf{a}_0^*\mathbf{a}_0 + \alpha + \beta + 1)(2\mathbf{a}_0^*\mathbf{a}_0 + \alpha + \beta + 2)^2(2\mathbf{a}_0^*\mathbf{a}_0 + \alpha + \beta + 3)}}\mathbf{a}_0\mathbf{a}_1^* \\ & + \mathbf{a}_0^*\mathbf{a}_1\sqrt{\frac{(2\mathbf{a}_0^*\mathbf{a}_0 + \mathbf{a}_1^*\mathbf{a}_1 + \alpha + \beta + 2)(\mathbf{a}_0^*\mathbf{a}_0 + \beta + 1)(\mathbf{a}_0^*\mathbf{a}_0 + \alpha + 1)(\mathbf{a}_0^*\mathbf{a}_0 + \alpha + \beta + 1)}{(2\mathbf{a}_0^*\mathbf{a}_0 + \alpha + \beta + 1)(2\mathbf{a}_0^*\mathbf{a}_0 + \alpha + \beta + 2)^2(2\mathbf{a}_0^*\mathbf{a}_0 + \alpha + \beta + 3)}}\end{aligned} \tag{5.23}$$

where $\alpha, \beta > -1$. The spectrum of $\mathbf{H}_I$ is

$$\sigma(\mathbf{H}_I) = \mathbb{N} \cup \{0\} \tag{5.24}$$

and the infinite-dimensional eigenspaces are spanned by the vectors

$$|E_{l,N}\rangle = \sum_{n=0}^{N} Q_n(E_{l,N}; \alpha, \beta, N)|n, N-n\rangle \qquad N = l+n, n = 0, 1, \ldots \tag{5.25}$$

where $E_{l,N} = l$ and

$$Q_n(E_{l,N}; \alpha, \beta, N) = \sqrt{\frac{(2n+\alpha+\beta+1)(\alpha+1)_n(-N)_n N!}{(-1)^n(n+\alpha+\beta+1)_{N+1}(\beta+1)_n n!}}\, {}_3F_2\left(\left.\begin{matrix}-n, n+\alpha+\beta+1, -l,\\ \alpha+1, -N\end{matrix}\right| 1\right) \tag{5.26}$$

are the Hahn polynomials. The weight function for the family $\{Q_n(E_{l,N}; \alpha, \beta, N)\}_{n=0}^{N}$ is

$$\frac{1}{\langle E_{l,N}|E_{l,N}\rangle} = \binom{\alpha+l}{l}\binom{\beta+N-l}{N-l}. \tag{5.27}$$

We can summarize that the spectral decomposition of the interaction Hamiltonian is

$$\mathbf{H}_I = \sum_{N=0}^{\infty}\sum_{l=0}^{N} l \frac{|E_{l,N}\rangle\langle E_{l,N}|}{\langle E_{l,N}|E_{l,N}\rangle}. \tag{5.28}$$

Let us note that putting $\alpha = \beta = 0$ we obtain the discrete Chebyshev polynomials.

5.5. *Integrable systems related to the dual q-Hahn polynomials*

For any fixed $0 < q < 1$ and $0 < \gamma, \delta < q^{-1}$ such that $0 < \gamma\delta < q^{-1}$ the Hamiltonian

$$\begin{aligned}\mathbf{H}_I = {} & 1 + \gamma\delta q - \gamma q(1 - q^{\mathbf{a}_0^*\mathbf{a}_0})(\delta - q^{-\mathbf{a}_1^*\mathbf{a}_1 - 1}) - (1 - q^{-\mathbf{a}_1^*\mathbf{a}_1})(1 - \gamma q^{\mathbf{a}_0^*\mathbf{a}_0+1}) \\ & + \sqrt{\frac{\gamma q(1 - q^{-\mathbf{a}_1^*\mathbf{a}_1})(1 - \gamma q^{\mathbf{a}_0^*\mathbf{a}_0+1})(1 - q^{\mathbf{a}_0^*\mathbf{a}_0+1})(\delta - q^{-\mathbf{a}_1^*\mathbf{a}_1})}{(\mathbf{a}_0^*\mathbf{a}_0+1)\mathbf{a}_1^*\mathbf{a}_1}}\mathbf{a}_0\mathbf{a}_1^* \\ & + \mathbf{a}_0^*\mathbf{a}_1\sqrt{\frac{\gamma q(1 - q^{-\mathbf{a}_1^*\mathbf{a}_1})(1 - \gamma q^{\mathbf{a}_0^*\mathbf{a}_0+1})(1 - q^{\mathbf{a}_0^*\mathbf{a}_0+1})(\delta - q^{-\mathbf{a}_1^*\mathbf{a}_1})}{(\mathbf{a}_0^*\mathbf{a}_0+1)\mathbf{a}_1^*\mathbf{a}_1}}\end{aligned} \tag{5.29}$$

has the spectrum

$$\sigma(\mathbf{H}_I) = \{q^{-l} + \gamma\delta q^{l+1} : l = 0, 1, \ldots\} \tag{5.30}$$

and the eigenspace $\mathcal{H}^l$ corresponding to eigenvalue $q^{-l} + \gamma\delta q^{l+1} \in \sigma(\mathbf{H}_I)$ is

$$\mathcal{H}^l = \mathrm{span}\{|E_{l,N}\rangle : N = l+n, n = 0, 1, \ldots\} \tag{5.31}$$

with

$$E_{l,N} = q^{-l} + \gamma\delta q^{l+1} \tag{5.32}$$

and

$$|E_{l,N}\rangle = \sum_{n=0}^{N} R_n(E_{l,N}; \gamma, \delta, N|q)|n, N-n\rangle \tag{5.33}$$

and

$$R_n(E_{l,N}; \gamma, \delta, N|q) = \sqrt{\frac{(\delta q; q)_N(\gamma q, q^{-N}; q)_n(\gamma q)^N}{(\gamma\delta q^2; q)_N(q, \delta^{-1}q^{-N}; q)_n(\gamma\delta q)^n}}\, {}_3\phi_2\left(\left.\begin{matrix}q^{-n}, q^{-l}, \gamma\delta q^{l+1}\\ \gamma q, q^{-N}\end{matrix}\right| q; q\right) \tag{5.34}$$

which are the dual q-Hahn polynomials. For each fixed N the family $\{R_n(E_{l,N};\gamma,\delta,N|q)\}_{n=0}^N$ forms an orthonormal system with respect to the weight function

$$\frac{1}{\langle E_{l,N}|E_{l,N}\rangle}=\frac{(\gamma q,\gamma\delta q,q^{-N};q)_l(1-\gamma\delta q^{2l+1})}{(q,\gamma\delta q^{N+2},\delta q;q)_l(1-\gamma\delta q)(-\gamma q)^l}q^{Nl-\binom{l}{2}}. \tag{5.35}$$

We can summarize that the spectral decomposition of the interaction Hamiltonian is

$$\mathbf{H}_I=\sum_{N=0}^{\infty}\sum_{l=0}^{N}(q^{-l}+\gamma\delta q^{l+1})\frac{|E_{l,N}\rangle\langle E_{l,N}|}{\langle E_{l,N}|E_{l,N}\rangle}. \tag{5.36}$$

5.6. Integrable systems related to the affine q-Krawtchouk polynomials

For any fixed $0<q<1$ and $0<p<q^{-1}$ the Hamiltonian

$$\begin{aligned}\mathbf{H}_I=1-[(1-q^{-\mathbf{a}_1^*\mathbf{a}_1})(1-pq^{\mathbf{a}_0^*\mathbf{a}_0+1})-pq^{-\mathbf{a}_1^*\mathbf{a}_1}(1-q^{\mathbf{a}_0^*\mathbf{a}_0})]\\+\sqrt{\frac{-pq^{-\mathbf{a}_1^*\mathbf{a}_1+1}(1-q^{\mathbf{a}_0^*\mathbf{a}_0+1})(1-q^{-\mathbf{a}_1^*\mathbf{a}_1})(1-pq^{\mathbf{a}_0^*\mathbf{a}_0+1})}{(\mathbf{a}_0^*\mathbf{a}_0+1)\mathbf{a}_1^*\mathbf{a}_1}}\mathbf{a}_0\mathbf{a}_1^*\\+\mathbf{a}_0^*\mathbf{a}_1\sqrt{\frac{-pq^{-\mathbf{a}_1^*\mathbf{a}_1+1}(1-q^{\mathbf{a}_0^*\mathbf{a}_0+1})(1-q^{-\mathbf{a}_1^*\mathbf{a}_1})(1-pq^{\mathbf{a}_0^*\mathbf{a}_0+1})}{(\mathbf{a}_0^*\mathbf{a}_0+1)\mathbf{a}_1^*\mathbf{a}_1}}\end{aligned} \tag{5.37}$$

has the spectrum

$$\sigma(\mathbf{H}_I)=\{q^{-l}:l=0,1,\ldots\}. \tag{5.38}$$

Eigenspaces $\mathcal{H}^l$, $q^{-l}\in\sigma(\mathbf{H}_I)$ are

$$\mathcal{H}^l=\mathrm{span}\{|E_{l,N}\rangle:N=l+n,n=0,1,\ldots\} \tag{5.39}$$

for $E_{l,N}=q^{-l}$,

$$|E_{l,N}\rangle=\sum_{n=0}^{N}K_n^{Aff}(E_{l,N};p,N;q)|n,N-n\rangle \tag{5.40}$$

and

$$K_n^{Aff}(E_{l,N};p,N;q)=\sqrt{\frac{(pq)^{N-n}(pq;q)_n(q;q)_N}{(q;q)_n(q;q)_{N-n}}}\,{}_3\phi_2\left(\begin{matrix}q^{-n},0,q^{-l}\\ pq,q^{-N}\end{matrix}\middle|\,q;q\right) \tag{5.41}$$

which are the affine q-Krawtchouk polynomials. The family $\left\{K_n^{Aff}(E_{l,N};p,N;q)\right\}_{n=0}^N$ is the orthonormal system with respect to the weight function

$$\frac{1}{\langle E_{l,N}|E_{l,N}\rangle}=\frac{(pq;q)_l(q;q)_N}{(q;q)_l(q;q)_{N-l}}(pq)^{-l}. \tag{5.42}$$

We can summarize that the spectral decomposition of the interaction Hamiltonian is

$$\mathbf{H}_I=\sum_{N=0}^{\infty}\sum_{l=0}^{N}q^{-l}\frac{|E_{l,N}\rangle\langle E_{l,N}|}{\langle E_{l,N}|E_{l,N}\rangle}. \tag{5.43}$$

5.7. Integrable systems related to the q-Krawtchouk polynomials

The Hamiltonian

$$\mathbf{H}_I = 1 - \frac{(1-q^{-\mathbf{a}_1^*\mathbf{a}_1})(1+pq^{\mathbf{a}_0^*\mathbf{a}_0})}{(1+pq^{2\mathbf{a}_0^*\mathbf{a}_0})(1+pq^{2\mathbf{a}_0^*\mathbf{a}_0+1})} + pq^{\mathbf{a}_0^*\mathbf{a}_0-\mathbf{a}_1^*\mathbf{a}_1-1}\frac{(1+pq^{2\mathbf{a}_0^*\mathbf{a}_0+\mathbf{a}_1^*\mathbf{a}_1})(1-q^{\mathbf{a}_0^*\mathbf{a}_0})}{(1+pq^{2\mathbf{a}_0^*\mathbf{a}_0-1})(1+pq^{2\mathbf{a}_0^*\mathbf{a}_0})}$$
$$+\sqrt{-\frac{pq^{\mathbf{a}_0^*\mathbf{a}_0-\mathbf{a}_1^*\mathbf{a}_1+1}(1+pq^{2\mathbf{a}_0^*\mathbf{a}_0+\mathbf{a}_1^*\mathbf{a}_1+1})(1-q^{\mathbf{a}_0^*\mathbf{a}_0+1})(1-q^{-\mathbf{a}_1^*\mathbf{a}_1})(1+pq^{\mathbf{a}_0^*\mathbf{a}_0})}{(1+pq^{2\mathbf{a}_0^*\mathbf{a}_0})(1+pq^{2\mathbf{a}_0^*\mathbf{a}_0+1})^2(1+pq^{2\mathbf{a}_0^*\mathbf{a}_0+2})(\mathbf{a}_0^*\mathbf{a}_0+1)\mathbf{a}_1^*\mathbf{a}_1}}\,\mathbf{a}_0\mathbf{a}_1^*$$
$$+\mathbf{a}_0^*\mathbf{a}_1\sqrt{-\frac{pq^{\mathbf{a}_0^*\mathbf{a}_0-\mathbf{a}_1^*\mathbf{a}_1+1}(1+pq^{2\mathbf{a}_0^*\mathbf{a}_0+\mathbf{a}_1^*\mathbf{a}_1+1})(1-q^{\mathbf{a}_0^*\mathbf{a}_0+1})(1-q^{-\mathbf{a}_1^*\mathbf{a}_1})(1+pq^{\mathbf{a}_0^*\mathbf{a}_0})}{(1+pq^{2\mathbf{a}_0^*\mathbf{a}_0})(1+pq^{2\mathbf{a}_0^*\mathbf{a}_0+1})^2(1+pq^{2\mathbf{a}_0^*\mathbf{a}_0+2})(\mathbf{a}_0^*\mathbf{a}_0+1)\mathbf{a}_1^*\mathbf{a}_1}} \tag{5.44}$$

where $0 < q < 1$ and $p > 0$, has the spectrum

$$\sigma(\mathbf{H}_I) = \{q^{-l} : l = 0, 1, \ldots\}. \tag{5.45}$$

Eigenspaces $\mathcal{H}^l$, $q^{-l} \in \sigma(\mathbf{H}_I)$ are

$$\mathcal{H}^l = \mathrm{span}\{|E_{l,N}\rangle : N = l + n, n = 0, 1, \ldots\} \tag{5.46}$$

where for $E_{l,N} = q^{-l}$

$$|E_{l,N}\rangle = \sum_{n=0}^{N} K_n(E_{l,N}; p, N; q)|n, N-n\rangle \tag{5.47}$$

and

$$K_n(E_{l,N}; p, N; q) = \sqrt{\frac{(-p, q^{-N}; q)_n(1+pq^{2n})p^N q^{\binom{N+1}{2}}}{(q, -pq^{N+1}; q)_n(1+p)(-pq; q)_N(-pq^{-N})^n q^{n^2}}}$$
$$\times\, {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{-l}, -pq^n \\ q^{-N}, 0 \end{matrix}\middle|\, q; q\right) \tag{5.48}$$

are the q-Krawtchouk polynomials. The weight function for the orthonormal system $\{K_n(E_{l,N}; p, N; q)\}_{n=0}^N$ is

$$\frac{1}{\langle E_{l,N}|E_{l,N}\rangle} = \frac{(q^{-N}; q)_l}{(q; q)_l}(-p)^{-l}. \tag{5.49}$$

We can summarize that the spectral decomposition of the interaction Hamiltonian is

$$\mathbf{H}_I = \sum_{N=0}^{\infty}\sum_{l=0}^{N} q^{-l}\frac{|E_{l,N}\rangle\langle E_{l,N}|}{\langle E_{l,N}|E_{l,N}\rangle}. \tag{5.50}$$

5.8. Integrable systems related to the q-Hahn polynomials

The Hamiltonian

$$\mathbf{H}_I = 1 + \frac{\alpha q^{-\mathbf{a}_1^*\mathbf{a}_1}(1-q^{\mathbf{a}_0^*\mathbf{a}_0})(1-\alpha\beta q^{2\mathbf{a}_0^*\mathbf{a}_0+\mathbf{a}_1^*\mathbf{a}_1+1})(1-\beta q^{\mathbf{a}_0^*\mathbf{a}_0})}{(1-\alpha\beta q^{2\mathbf{a}_0^*\mathbf{a}_0})(1-\alpha\beta q^{2\mathbf{a}_0^*\mathbf{a}_0+1})}$$
$$-\frac{(1-q^{-\mathbf{a}_1^*\mathbf{a}_1})(1-\alpha q^{\mathbf{a}_0^*\mathbf{a}_0+1})(1-\alpha\beta q^{\mathbf{a}_0^*\mathbf{a}_0+1})}{(1-\alpha\beta q^{2\mathbf{a}_0^*\mathbf{a}_0+2})(1-\alpha\beta q^{2\mathbf{a}_0^*\mathbf{a}_0+1})}$$
$$+\sqrt{-\frac{\alpha q^{-\mathbf{a}_1^*\mathbf{a}_1+1}(1-q^{\mathbf{a}_0^*\mathbf{a}_0+1})(1-\alpha\beta q^{2\mathbf{a}_0^*\mathbf{a}_0+\mathbf{a}_1^*\mathbf{a}_1+2})(1-\beta q^{\mathbf{a}_0^*\mathbf{a}_0+1})}{(1-\alpha\beta q^{2\mathbf{a}_0^*\mathbf{a}_0+1})(1-\alpha\beta q^{2\mathbf{a}_0^*\mathbf{a}_0+2})^2(1-\alpha\beta q^{2\mathbf{a}_0^*\mathbf{a}_0+3})(\mathbf{a}_0^*\mathbf{a}_0+1)\mathbf{a}_1^*\mathbf{a}_1}}$$

$$\times \sqrt{(1-q^{-\mathbf{a}_1^*\mathbf{a}_1})(1-\alpha q^{\mathbf{a}_0^*\mathbf{a}_0+1})(1-\alpha\beta q^{\mathbf{a}_0^*\mathbf{a}_0+1})}\mathbf{a}_0\mathbf{a}_1^*$$
$$+\mathbf{a}_0^*\mathbf{a}_1\sqrt{-\frac{\alpha q^{-\mathbf{a}_1^*\mathbf{a}_1+1}(1-q^{\mathbf{a}_0^*\mathbf{a}_0+1})(1-\alpha\beta q^{2\mathbf{a}_0^*\mathbf{a}_0+\mathbf{a}_1^*\mathbf{a}_1+2})(1-\beta q^{\mathbf{a}_0^*\mathbf{a}_0+1})}{(1-\alpha\beta q^{2\mathbf{a}_0^*\mathbf{a}_0+1})(1-\alpha\beta q^{2\mathbf{a}_0^*\mathbf{a}_0+2})^2(1-\alpha\beta q^{2\mathbf{a}_0^*\mathbf{a}_0+3})(\mathbf{a}_0^*\mathbf{a}_0+1)\mathbf{a}_1^*\mathbf{a}_1}}$$
$$\times \sqrt{(1-q^{-\mathbf{a}_1^*\mathbf{a}_1})(1-\alpha q^{\mathbf{a}_0^*\mathbf{a}_0+1})(1-\alpha\beta q^{\mathbf{a}_0^*\mathbf{a}_0+1})} \tag{5.51}$$

where $0 < q < 1$, $0 < \alpha < q^{-1}$ and $0 < \beta < q^{-1}$ has the spectrum

$$\sigma(\mathbf{H}_I) = \{q^{-l} : l = 0, 1, \ldots\}. \tag{5.52}$$

Eigenspaces $\mathcal{H}^l$, $q^{-l} \in \sigma(\mathbf{H}_I)$ are

$$\mathcal{H}^l = \mathrm{span}\{|E_{l,N}\rangle : N = l + n, n = 0, 1, \ldots\} \tag{5.53}$$

where $E_{l,N} = q^{-l}$

$$|E_{l,N}\rangle = \sum_{n=0}^{N} Q_n(E_{l,N}; \alpha, \beta, N|q)|n, N-n\rangle \tag{5.54}$$

and

$$Q_n(E_{l,N}; \alpha, \beta, N|q) = \sqrt{\frac{(\beta q; q)_N(\alpha q)^N}{(\alpha\beta q^2; q)_N}\frac{(\alpha q, \alpha\beta q, q^{-N}; q)_n(1-\alpha\beta q^{2n+1})q^{Nn-\binom{n}{2}}}{(q, \alpha\beta q^{N+2}, \beta q; q)_n(1-\alpha\beta q)(-\alpha q)^n}}$$
$$\times {}_3\phi_2\left(\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-l} \\ \alpha q, q^{-N}\end{matrix}\middle| q; q\right) \tag{5.55}$$

are the q-Hahn polynomials. The appropriate weight function is

$$\frac{1}{\langle E_{l,N}|E_{l,N}\rangle} = \frac{(\alpha q, q^{-N}; q)_l}{(q, \beta^{-1}q^{-N}; q)_l}(\alpha\beta q)^{-l}. \tag{5.56}$$

We can summarize that the spectral decomposition of the interaction Hamiltonian is

$$\mathbf{H}_I = \sum_{N=0}^{\infty}\sum_{l=0}^{N} q^{-l}\frac{|E_{l,N}\rangle\langle E_{l,N}|}{\langle E_{l,N}|E_{l,N}\rangle}. \tag{5.57}$$

5.9. Integrable systems related to the dual q-Krawtchouk polynomials

The Hamiltonian

$$\mathbf{H}_I = (1+c)q^{-\mathbf{a}_1^*\mathbf{a}_1}$$
$$+\sqrt{\frac{cq^{-(\mathbf{a}_0^*\mathbf{a}_0+\mathbf{a}_1^*\mathbf{a}_1)}(1-q^{-\mathbf{a}_1^*\mathbf{a}_1})(1-q^{\mathbf{a}_0^*\mathbf{a}_0+1})}{(\mathbf{a}_0^*\mathbf{a}_0+1)\mathbf{a}_1^*\mathbf{a}_1}}\mathbf{a}_0\mathbf{a}_1^*$$
$$+\mathbf{a}_0^*\mathbf{a}_1\sqrt{\frac{cq^{-(\mathbf{a}_0^*\mathbf{a}_0+\mathbf{a}_1^*\mathbf{a}_1)}(1-q^{-\mathbf{a}_1^*\mathbf{a}_1})(1-q^{\mathbf{a}_0^*\mathbf{a}_0+1})}{(\mathbf{a}_0^*\mathbf{a}_0+1)\mathbf{a}_1^*\mathbf{a}_1}} \tag{5.58}$$

where $0 < q < 1$, $c < 0$ has the spectrum

$$\sigma(\mathbf{H}_I) = \{q^{-l} + cq^{l-N} : l = 0, 1, \ldots N, N = 0, 1, \ldots\}. \tag{5.59}$$

The eigenvalues $E_{l,N} = q^{-l} + cq^{l-N}$, $l = 0, 1, \ldots, N$ of the reduced Hamiltonian $\mathbf{H}_N$ depend on N, and therefore the eigenspaces of $\mathbf{H}_I$ are one dimensional given by the vectors

$$|E_{l,N}\rangle = \sum_{n=0}^{N} K_n(E_{l,N}; c, N|q)|n, N-n\rangle \tag{5.60}$$

where the coefficients

$$K_n(E_{l,N}; c, N|q) = \sqrt{\frac{(q^{-N}; q)_n}{(c^{-1}; q)_N (q; q)_n (cq^{-N})^n}}\; {}_3\phi_2 \left(\begin{matrix} q^{-n}, q^{-l}, cq^{l-N} \\ q^{-N}, 0 \end{matrix} \middle| q; q \right) \tag{5.61}$$

are the dual q-Krawtchouk polynomials. For each fixed N the finite family $\{K_n(E_{l,N}; c, N|q)\}_{n=0}^{N}$ is an orthonormal system with respect to the weight function

$$\frac{1}{\langle E_{l,N}|E_{l,N}\rangle} = \frac{(cq^{-N}, q^{-N}; q)_l (1 - cq^{2l-N})}{(q, cq; q)_l (1 - cq^{-N})} c^{-l} q^{l(2N-l)}. \tag{5.62}$$

We can summarize that the spectral decomposition of the interaction Hamiltonian is

$$\mathbf{H}_I = \sum_{N=0}^{\infty} \sum_{l=0}^{N} (q^{-l} + cq^{l-N}) \frac{|E_{l,N}\rangle\langle E_{l,N}|}{\langle E_{l,N}|E_{l,N}\rangle}. \tag{5.63}$$

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